

# Hydrodynamic Limit for an Hamiltonian system with Boundary Conditions and Conservative Noise

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## Abstract

We study the hyperbolic scaling limit for a chain of  $N$  coupled anharmonic oscillators. The chain is open and with the following adiabatic boundary conditions: it is attached to a wall on the left and there is a force (tension)  $\tau$  acting on the right. In order to provide the system of the good ergodic properties, we perturb the Hamiltonian dynamics with random local exchanges of velocities between the particles, so that momentum and energy are locally conserved. We prove that in the macroscopic limit the distribution of the density of particles, momentum and energy converge to the solution of the Euler equations, in the smooth regime of them.

## 1 Introduction

The aim of this paper is to study the hydrodynamic limit for a non-equilibrium system subject to an exterior time dependent force at the boundary. We consider the most simple mechanical model with non-linear interaction, i.e. a one dimensional chain of  $N$  anharmonic oscillators. The left side is attached to a wall, while on the right side is acting a force  $\tau$  (tension). For each value of  $\tau$  there is a family of equilibrium (Gibbs) measures parametrized by the temperature (and by the tension  $\tau$ ). It turns out for this specific model that these Gibbs measures can be written as a product.

We are interested in the macroscopic non-equilibrium behaviour of this system as  $N$  tends to infinity, after rescaling space and time with  $N$  in the same way (*hyperbolic scaling*). We also consider situations in which the tension  $\tau$  depends slowly on time, such that it changes in the macroscopic time scale. In this way we can also take the system originally at equilibrium at a certain tension  $\tau_0$  and push out of equilibrium by changing the exterior tension.

The goal is to prove that the 3 main conserved quantities (density, momentum and energy) satisfy in the limit an autonomous closed set of hyperbolic equations given by the Euler system.

We approach this problem by using the *relative entropy method* (cf. [10]) as already done in [8] for a system of interacting particles moving in  $\mathbb{R}^3$  (*gaz dynamics*).

The relative entropy method permits, in general, to obtain such hydrodynamic limit if the system satisfy certain conditions:

- A) The dynamic should be *ergodic* in the sense that the only conserved quantities that *survive* the limit as  $N \rightarrow \infty$  are those we are looking for the macroscopic autonomous behavior (in this case density, momentum and energy). More precisely, the only stationary measure for the infinite system, with finite local entropy, are given by the Gibbs measures.

- B) The limit can be obtained in the smooth regime of the macroscopic equation.
- C) Microscopic currents of the conserved quantities should be bounded by the energy of the system.

We do not know any deterministic hamiltonian system that satisfy condition A, and this is a major challenging open problem in statistical mechanics. Stochastic perturbation of the dynamics that conserves energy and momentum can give such ergodic property and have been used in [8] (cf. also [7, 3, 2]). We use here a simpler stochastic mechanism than in [8]: at random independent exponential times we exchange the momentum of nearest neighbor particles, as if they were performing an elastic collision. Under this stochastic dynamics, every stationary measure has the property to be exchangeable in the velocity coordinates, and this is sufficient to characterize it as a convex combination of Gibbs measures (cf. [2] and [1]).

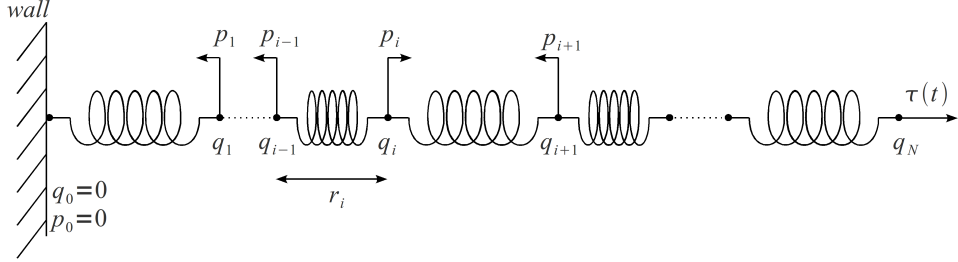
About condition B, it is well known that nonlinear hyperbolic equations in general develop shocks also starting from smooth initial condition. Characterization and uniqueness of weak solutions in presence of shock is a challenging problem in the theory of hyperbolic equations. We expect shock decrease the thermodynamic entropy associated to the profiles of the conserved quantities (i.e. a macroscopic decrease of the thermodynamic entropy). The way relative entropy work in this case is in comparing the microscopic Gibbs entropy production (associated to the probability distribution of the system at a given time) with the macroscopic (thermodynamic) entropy production. If no shocks are present both entropy productions are small. The presence of the boundary force changes a bit this balance, since one should take into account the (macroscopic) change of entropy due to the work performed by the force. It turns out that the right choice of the boundary conditions in the macroscopic equation compensate this large entropy production, keeping the time derivative of the relative entropy small. It would be interesting to prove similar cancellation of entropy productions when this is caused by shocks, as it would allow to prove the hydrodynamical limit in these cases. Recent efforts in this direction use different methods (cf. [4]).

About condition C, it created a problem in [8], since in the usual gaz dynamics the energy current has the convecting term cubic in the velocities, while energy is quadratic. This was fixed in [8] by modifying the kinetic energy of the model: if the kinetic energy grows linearly as a function of the velocity, the energy current will grow also linearly. Since we work here in lagrangian coordinates, our energy current does not have the cubic convecting term. This allows us to work with the usual quadratic kinetic energy.

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## 2 The Model and the Main Theorem

We will study a system of  $N$  coupled oscillators in one dimension. Each particle has the same mass that we set equal to 1. The position of atom  $i$  is denoted by  $q_i \in \mathbb{R}$ , while its momentum is denoted by  $p_i \in \mathbb{R}$ . Thus the configuration space is  $(\mathbb{R} \times \mathbb{R})^N$ . We assume that an extra particle 0 to be attached to a wall and does not move, i.e.  $(q_0, p_0) \equiv (0, 0)$ , while on particle  $N$  we apply a force  $\tau(t)$  depending on time.



Denote by  $\mathbf{q} := (q_1, \dots, q_N)$  and  $\mathbf{p} := (p_1, \dots, p_N)$ . The interaction between two particles  $i$  and  $i - 1$  will be described by the potential energy  $V(q_i - q_{i-1})$  of an anharmonic spring relying the particles. We assume  $V$  to be a positive smooth function which for large  $r$  grows faster than linear but at most quadratic, that means that there exists a constant  $C > 0$  such that

$$\lim_{|r| \rightarrow \infty} \frac{V(r)}{|r|} = \infty. \quad (2.1)$$

$$\limsup_{|r| \rightarrow \infty} V''(r) \leq C < \infty. \quad (2.2)$$

Energy is defined by the following Hamiltonian:

$$\mathcal{H}_N(\mathbf{q}, \mathbf{p}) : = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=1}^N V(q_i - q_{i-1}).$$

Since we focus on a nearest neighbor interaction, we may define the distance between particles by

$$r_i = q_i - q_{i-1}, \quad i = 1, \dots, N.$$

We define the energy of particle  $i \in \{1, \dots, N\}$  as

$$e_i := \frac{p_i^2}{2} + V(r_i)$$

so that  $\mathcal{H}_N(\mathbf{r}, \mathbf{p}) = \sum_{i=1}^N e_i$ , where  $\mathbf{r} := (r_1, \dots, r_N)$ .

The dynamics of the system is determined by the generator

$$\mathcal{G}_N := L_N^\tau + \gamma S_N. \quad (2.3)$$

Here the Liouville operator  $L_N^\tau$  is given by

$$\begin{aligned} L_N^\tau &= \sum_{i=1}^N (p_i - p_{i-1}) \frac{\partial}{\partial r_i} + \sum_{i=1}^{N-1} (V'(r_{i+1}) - V'(r_i)) \frac{\partial}{\partial p_i} \\ &\quad + (\tau(t) - V'(r_N)) \frac{\partial}{\partial p_N}, \end{aligned} \quad (2.4)$$

where we used the fact that  $p_0 \equiv 0$ .

The symmetric operator  $S_N$  is the generator of the stochastic part of the dynamics that exchange at random time velocities of nearest neighbor particles. For any smooth function  $f$ , we define the operator  $\Upsilon_{i,i+1}$  by

$$\Upsilon_{i,i+1} = \frac{1}{2} \left( f(\mathbf{r}, \mathbf{p}^{i,i+1}) - f(\mathbf{r}, \mathbf{p}) \right) \quad (2.5)$$

where  $\mathbf{p}^{i,i+1} \in \mathbb{R}^N$  is defined from  $\mathbf{p} \in \mathbb{R}^N$  by exchanging the coordinates  $p_j$  and  $p_{j+1}$

$$p_j^{i,i+1} = \begin{cases} p_j & \text{if } j \neq i, i+1 \\ p_{i+1} & \text{if } j = i \\ p_i & \text{if } j = i+1 \end{cases}.$$

Then  $S_N$  is defined through

$$S_N f(\mathbf{r}, \mathbf{p}) := \sum_{i=1}^{N-1} \left( f(\mathbf{r}, \mathbf{p}^{i,i+1}) - f(\mathbf{r}, \mathbf{p}) \right) \quad (2.6)$$

$$= - \sum_{i=1}^{N-1} \Upsilon_{i,i+1}^2 f(\mathbf{r}, \mathbf{p}) = 2 \sum_{i=1}^{N-1} \Upsilon_{i,i+1} f(\mathbf{r}, \mathbf{p}), \quad (2.7)$$

With this choice of the noise, the three balanced quantities i.e. locally conserved, are given by  $r_i, p_i, e_i$ .

We define  $\zeta(r, p) = (r, p, -e(r, p)) \in \mathbb{R}^2 \times \mathbb{R}_-$ , and the partition function  $Z$  on  $\mathbb{R}^2 \times \mathbb{R}_+$  by

$$Z(\boldsymbol{\lambda}) = Z(\lambda_1, \lambda_2, \lambda_3) := \int_{\mathbb{R}^2} e^{\boldsymbol{\lambda} \cdot \zeta(r, p)} dr dp.$$

and the free energy function as its logarithm:

$$\Theta(\boldsymbol{\lambda}) := \log Z(\boldsymbol{\lambda}), \quad (2.8)$$

By the condition imposed on  $V$ , this function is always finite.

For  $\zeta \in \mathbb{R}^2 \times \mathbb{R}_-$  we define  $\Phi : \mathbb{R}^2 \times \mathbb{R}_- \rightarrow \mathbb{R}$  by the Legendre transform of the free energy function

$$\Phi(\zeta) := \sup_{\boldsymbol{\eta} \in \mathbb{R}^2 \times \mathbb{R}_+} \{ \boldsymbol{\eta} \cdot \zeta - \Theta(\boldsymbol{\eta}) \}. \quad (2.9)$$

So that the inverse is

$$\Theta(\boldsymbol{\eta}) := \sup_{\zeta \in \mathbb{R}^2 \times \mathbb{R}_-} \{ \boldsymbol{\eta} \cdot \zeta(r, p) - \Phi(\zeta) \}. \quad (2.10)$$

We denote by  $\boldsymbol{\lambda}(\tilde{\mathbf{u}})$  and  $\tilde{\mathbf{u}}(\boldsymbol{\lambda}) := (\mathbf{r}, \mathbf{p}, -E)$  the corresponding convex conjugate variable, that satisfy

$$\boldsymbol{\lambda} = D\Phi(\tilde{\mathbf{u}}) \quad \text{and} \quad \tilde{\mathbf{u}} = D\Theta(\boldsymbol{\lambda}), \quad (2.11)$$

where the operator  $D$  is defined by

$$Df(\mathbf{a}) := \left( \frac{\partial f}{\partial a_1}, \frac{\partial f}{\partial a_2}, \frac{\partial f}{\partial a_3} \right)^T \quad (2.12)$$

for any  $C^1$  function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $\mathbf{a} := (a_1, a_2, a_3) \in \mathbb{R}^3$ .

On the one particle state space  $\mathbb{R}^2$  we define a family of probability measure

$$\nu_{\boldsymbol{\lambda}}(dr, dp) = e^{\boldsymbol{\lambda} \cdot \zeta(r, p) - \Theta(\boldsymbol{\lambda})} dr dp \quad (2.13)$$

Observe that

$$E_{\nu_{\boldsymbol{\lambda}}}[\zeta(r, p)] = \tilde{\mathbf{u}}$$

so we can identify  $\tilde{\mathbf{u}} = (\mathbf{r}, \mathbf{p}, -E)$  as respectively the average distance, velocity and (negative) energy. We also define the *internal energy*  $\mathfrak{e} = E - \mathbf{p}^2/2$ , and it is easy to see that  $\boldsymbol{\lambda}$  is actually a function of  $\mathbf{r}, \mathbf{p}, \mathfrak{e}$ . We have also the relations

$$E_{\nu_{\boldsymbol{\lambda}}}(p^2) - \mathbf{p}^2 = \lambda_3^{-1} := \beta^{-1}, \quad P(\mathbf{r}, \mathfrak{e}) := E_{\nu_{\boldsymbol{\lambda}}}[V'(r)] = \frac{\lambda_1}{\lambda_3} := \tau$$

that identify  $\beta^{-1}$  as temperature and  $\tau$  as tension. This thermodynamic terminology is justified by observing that, for constant  $\tau$  in the dynamics, and any  $\beta > 0$ , with the choice  $\boldsymbol{\lambda} = (\beta\tau, 0, \beta)$  the family of product measures given by:

$$\nu_{(\tau\beta, 0, \beta)}^N(d\mathbf{r}, d\mathbf{p}) = \prod_{i=1}^N \nu_{(\tau\beta, 0, \beta)}(dr_i, dp_i), \quad \beta \in \mathbb{R}^+$$

is stationary for the dynamics. These are the grand canonical Gibbs measures at an average temperature  $\beta^{-1}$ , pressure  $\tau$  and velocity 0.

In what follows we need also Gibbs measure with average velocity different from 0, and we will use the following notation:

$$\nu_{\boldsymbol{\lambda}}^N := \prod_{i=1}^N e^{\boldsymbol{\lambda} \cdot \boldsymbol{\zeta}_i - \Theta(\boldsymbol{\lambda})} dr_i dp_i,$$

where  $\boldsymbol{\zeta}_i := (\zeta_{i,1}, \zeta_{i,2}, \zeta_{i,3}) := (r_i, p_i, -e_i)$ .

In a similar way we may introduce the local Gibbs measures: For any continuous profile  $\tilde{u}(x)$ ,  $x \in [0, 1]$ , we have correspondingly a *profile* of parameters  $\boldsymbol{\lambda}(x)$ , and we define the inhomogeneous product measure

$$\nu_{\boldsymbol{\lambda}(\cdot)}^N := \prod_{i=1}^N e^{\boldsymbol{\lambda}(i/N) \cdot \boldsymbol{\zeta}_i - \Theta(\boldsymbol{\lambda}(i/N))} dr_i dp_i, \quad (2.14)$$

that we call *Local Gibbs measures*.

We are interested in the macroscopic behavior of the interdistance, momentum and energy of the particles, at time  $Nt$ , as  $N \rightarrow \infty$ . Taking advantage of the one-dimensionality of the system, we will use *lagrangian* coordinates, i.e. our space variables will be given by the lattice coordinates  $\{1/N, \dots, (N-1)/N, 1\}$ .

Consequently, we introduce the (time dependent) empirical measures representing the spatial distribution (on the interval  $[0, 1]$ ) of these quantities:

$$\eta_{\alpha}^N(dx, t) := \frac{1}{N} \sum_{i=1}^N \delta\left(x - \frac{i}{N}\right) \zeta_{i,\alpha}(Nt) dx, \quad \text{for } \alpha = 1, 2, 3.$$

We expect the measures  $\eta_{\alpha}^N(dx, t)$ ,  $\alpha = 1, 2, 3$  to converge, as  $N \rightarrow \infty$ , to measures  $\mathbf{r}(x, t)dx$ ,  $\mathbf{p}(x, t)dx$ ,  $-E(x, t)dx$  being absolutely continuous with respect to the Lebesgue measure and with density satisfying the following system of three conservation laws:

$$\begin{cases} \partial_t \mathbf{r} - \partial_x \mathbf{p} = 0 \\ \partial_t \mathbf{p} - \partial_x P(\mathbf{r}, \mathbf{e}) = 0 \\ \partial_t E - \partial_x (\mathbf{p} P(\mathbf{r}, \mathbf{e})) = 0 \end{cases}, \quad \begin{cases} \mathbf{r}_0(x) = \mathbf{r}(x, 0), \mathbf{p}_0(x) = \mathbf{p}(x, 0), E_0(x) = E(x, 0) \\ \mathbf{p}(0, t) = 0, P(\mathbf{r}(1, t), \mathbf{e}(1, t)) = \tau(t) \end{cases}, \quad (2.15)$$

for bounded, smooth initial data  $\mathbf{r}_0, \mathbf{p}_0, E_0 : [0, 1] \rightarrow \mathbb{R}$  and the force  $\tau(t)$  depending on time  $t$ . Here we denoted by  $\mathbf{r}$  the specific volume,  $\mathbf{p}$  the velocity,  $E$  the total energy and  $\mathbf{e} := E - \frac{1}{2}\mathbf{p}^2$  the internal energy.

We need the solutions of the system (2.15) to be  $C^2$ -solutions. To assure this, the following additional compatibility conditions at the space-time edges  $(x, t) = (0, 0)$  and  $(x, t) = (1, 0)$  have to be satisfied:

$$\lim_{x \rightarrow 0} \mathbf{p}_0(x) = \mathbf{p}(0, 0) = 0, \quad \lim_{x \rightarrow 1} P(\mathbf{r}_0(x), \mathbf{e}_0(x)) = \tau(0) \quad (2.16)$$

$$\lim_{x \rightarrow 0} \frac{d}{dx} P(\mathbf{r}_0(x), \mathbf{e}_0(x)) = 0, \quad \lim_{x \rightarrow 1} \frac{d}{dt} P(\mathbf{r}_0(x), \mathbf{e}_0(x)) = \tau'(0) \quad (2.17)$$

$$\lim_{x \rightarrow 0} \frac{d^2}{(dt)^2} \mathbf{p}_0(x) = 0, \quad \lim_{x \rightarrow 1} \frac{d^2}{(dt)^2} P(\mathbf{r}_0(x), \mathbf{e}_0(x)) = \tau''(0). \quad (2.18)$$

A proof of this can be adapted from Chapter 3.5 and 7 of [6].

For any test function  $J : [0, 1] \rightarrow \mathbb{R}$  with compact support in  $(0, 1)$  consider the empirical densities

$$\eta_\alpha^N(t, J) := \langle \eta_\alpha^N(dx, t); J \rangle = \frac{1}{N} \sum_{i=1}^N J\left(\frac{i}{N}\right) \zeta_{\alpha,i}(Nt). \quad (2.19)$$

Our goal is to show that, starting with an initial distribution such that there exist smooth functions  $\mathbf{r}_0$ ,  $\mathbf{p}_0$  and  $E_0$  satisfying

$$\{\eta_1^N(0, J), \eta_2^N(0, J), \eta_3^N(0, J)\} \rightarrow \left\{ \int J(x) \mathbf{r}_0(x) dx, \int J(x) \mathbf{p}_0(x) dx, - \int J(x) E_0(x) dx \right\} \quad (2.20)$$

in probability as  $N \rightarrow \infty$ , then at time  $t \in [0, T]$  we have the same convergence of  $\eta_\alpha^N(t, J)$ ,  $\alpha = 1, 2, 3$  to the corresponding profiles  $\mathbf{r}(x, t)$ ,  $\mathbf{p}(x, t)$  and  $E(x, t)$  respectively, that satisfy (2.15)–(2.18).

Here is the precise statement of our main result, where we make a stronger assumption on the initial measure:

**Theorem 2.1** (Main Theorem). *For any time  $t \in [0, T]$ , denote by  $\mu_t^N$  the probability measure on the path space  $C([0, T], (\mathbb{R}^2)^N)$  of our process with generator  $N\mathcal{G}_N$ , and starting from the local Gibbs measure  $\nu_{\lambda^{(c,0)}}^N$  corresponding to the initial profiles  $\tilde{\mathbf{u}}_0$ . Then for any smooth function  $J : [0, 1] \rightarrow \mathbb{R}$  and any  $\delta > 0$*

$$\lim_{N \rightarrow \infty} \nu_t^N \left[ \left| \frac{1}{N} \sum_{i=1}^N J\left(\frac{i}{N}\right) \zeta_i - \int_0^1 J(x) \tilde{\mathbf{u}}(x, t) dx \right| > \delta \right] = 0. \quad (2.21)$$

where  $\mathbf{u}$  is a  $C^2$ -solution to the system of conservation laws (2.15)–(2.18) and  $0 < T < t_s$ ,  $t_s$  being the time at which the solution  $\mathbf{u}$  produces the first shock.

**Remark 2.2.** As our proof is based on the relative entropy method of [10], it is only valid as long as the solution to (2.15) are  $C^2$ . Since, even for smooth initial data, the solution will develop shocks, we are forced to restrict our derivation to a time  $0 < T < t_s$ , where  $t_s$  is the time when the solution to the system of conservation laws enters the first shock.

**Remark 2.3.** A proof for the existence of smooth solutions to the initial-boundary-value problem (2.15) can be found in [6]. Notice that we can rewrite the pressure  $P$  as a function of specific volume  $\mathbf{r}$  and entropy  $\mathbf{s}$ :

$$\tilde{P}(\mathbf{r}, \mathbf{s}) := P(\mathbf{r}, \mathbf{e}).$$

Then we can rewrite the initial boundary value problem (2.15), in the smooth regime, in terms of the unknown  $\mathbf{r}$ ,  $\mathbf{p}$  and  $\mathbf{s}(\mathbf{r}, \mathbf{e})$  as follows:

$$\begin{cases} \partial_t \mathbf{r} - \partial_x \mathbf{p} = 0 \\ \partial_t \mathbf{p} - \partial_x \tilde{P}(\mathbf{r}, \mathbf{s}) = 0 \\ \partial_t \mathbf{s} = 0 \end{cases}, \begin{cases} \mathbf{r}_0(x) = \mathbf{r}(x, 0), \mathbf{p}_0(x) = \mathbf{p}(x, 0), \mathbf{s}_0(x) = \mathbf{s}(x, 0) \\ \mathbf{p}(0, t) = 0, \tilde{P}(\mathbf{r}(1, t), \mathbf{s}(1, t)) = \tau(t) \end{cases}, \quad (2.22)$$

where we used the thermodynamic relation

$$\tilde{P}(\mathbf{r}, \mathbf{s}) = - \frac{\partial \mathbf{e}(\mathbf{r}, \mathbf{s})}{\partial \mathbf{r}}.$$

Hence the specific entropy  $\mathbf{s}$  does not change in time and for any  $x \in [0, 1]$  is given through the initial data  $\mathbf{s}(x, 0) := \mathbf{s}_0(x)$ .

In the non-consevative form, equation (2.22) reads as:

$$\partial_t \begin{pmatrix} \mathbf{r} \\ \mathbf{p} \\ \mathbf{s} \end{pmatrix} - \mathbf{A}(\mathbf{r}, \mathbf{p}, \mathbf{s}) \partial_x \begin{pmatrix} \mathbf{r} \\ \mathbf{p} \\ \mathbf{s} \end{pmatrix} = 0$$

where the  $3 \times 3$ -matrix  $\mathbf{A}$  is defined by

$$\mathbf{A} := \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial \tilde{P}}{\partial \mathbf{r}} & 0 & \frac{\partial \tilde{P}}{\partial \mathbf{s}} \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{S} \cdot \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \mathbf{S}^{-1}$$

with  $c := c(\mathbf{r}, \mathbf{s}) = \sqrt{\frac{\partial \tilde{P}}{\partial \mathbf{r}}}$  and

$$\mathbf{S} := \mathbf{S}(\mathbf{r}, \mathbf{p}, \mathbf{s}) = \begin{pmatrix} 1 & 1 & -\frac{1}{c} \frac{\partial \tilde{P}}{\partial \mathbf{s}} \\ c & -c & 0 \\ 0 & 0 & c \end{pmatrix}.$$

With these notations we can rewrite (2.22) in the characteristic form

$$\begin{aligned} \mathbf{S}^{-1} \cdot \partial_t \begin{pmatrix} \mathbf{r} \\ \mathbf{p} \\ \mathbf{s} \end{pmatrix} - \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \mathbf{S}^{-1} \cdot \partial_x \begin{pmatrix} \mathbf{r} \\ \mathbf{p} \\ \mathbf{s} \end{pmatrix} &= 0 \\ \Rightarrow \begin{cases} c(\partial_t \mathbf{r} - c \partial_x \mathbf{r}) + (\partial_t \mathbf{p} - c \partial_x \mathbf{p}) + \frac{1}{c} \frac{\partial \tilde{P}}{\partial \mathbf{s}} (\partial_t \mathbf{s} - c \partial_x \mathbf{s}) = 0 \\ c(\partial_t \mathbf{r} + c \partial_x \mathbf{r}) - (\partial_t \mathbf{p} + c \partial_x \mathbf{p}) + \frac{1}{c} \frac{\partial \tilde{P}}{\partial \mathbf{s}} (\partial_t \mathbf{s} + c \partial_x \mathbf{s}) = 0 \\ \partial_t \mathbf{s} = 0. \end{cases} \end{aligned}$$

In this way we can apply the existence proof for  $C^2$  solutions to (2.15) for short times from [6].

## 3 The Hydrodynamic Limit

### 3.1 The Relative Entropy

On the phase space  $(\mathbb{R}^2)^N$  we now have two time dependent families of probability measures. One of them is the local Gibbs measure  $\nu_{\lambda(\cdot, t)}^N$  constructed from the solution of system of conservation laws (2.15)–(2.18). We denote its density by

$$g_t^N = \prod_{i=1}^N e^{\lambda(i/N, t) \cdot \zeta_i - \Theta(\lambda(i/N, t))} \quad (3.1)$$

On the other hand we have the actual distribution, whose density  $f_t^N(\mathbf{r}, \mathbf{p})$  is the solution of the Kolmogorov equation:

$$\begin{cases} \frac{\partial f_t^N}{\partial t}(\mathbf{r}, \mathbf{p}) &= N \mathcal{G}_N^* f_t^N(\mathbf{r}, \mathbf{p}) \\ f_0^N(\mathbf{r}, \mathbf{p}) &= g_0^N(\mathbf{r}, \mathbf{p}). \end{cases} \quad (3.2)$$

By  $\mathcal{G}_N^* = L_N^{\tau, * *} + \gamma S_N$  we denote the adjoint operator of  $\mathcal{G}_N$ , which can be computed as  $L_N^{\tau, *} = -L_N^\tau$ .

The relative entropy of  $f_t^N$  with respect to  $g_t^N$  is defined by

$$H_N(t) = \int f_t^N \log \frac{f_t^N}{g_t^N} d\mathbf{r} d\mathbf{p} \quad (3.3)$$

Our main result will follow from:

**Theorem 3.1** (Relative entropy). *Under the same assumptions as in Theorem 2.1, for any time  $t \in [0, T]$*

$$\lim_{N \rightarrow \infty} \frac{1}{N} H_N(t) = 0.$$

Notice that the relative entropy  $H(\alpha|\beta)$  of a probability measure  $\alpha$  with respect to a reference measure  $\beta$  and having densities  $f$  and  $g$  respectively with respect to the Lebesgue measure, can be rewritten as

$$H(\alpha|\beta) = \sup_{\varphi} \left\{ \int \varphi d\alpha - \log \int e^{\varphi} d\beta \right\} \quad (3.4)$$

where the supremum is taken over all bounded functions  $\varphi$ . It is easy to see that the relative entropy has the following properties:  $H(\alpha|\beta)$  is positive, convex and lower semicontinuous. It follows that for any measurable function  $F$ , any positive constant  $\sigma$  and some probability measures  $\alpha$  and  $\beta$ :

$$E_{\alpha}[F] \leq \frac{1}{\sigma} \log E_{\beta}[\exp(\sigma F)] + \frac{1}{\sigma} H(\alpha|\beta). \quad (3.5)$$

To see how Theorem 3.1 implies the Main Theorem, please see [5, 1]

### 3.2 Time Evolution of the Relative Entropy

In this Section we will prove Theorem 3.1.

Notice that with the choice of our initial distribution the relative entropy at time 0 is equal to zero:  $H_N(0) = 0$ . The strategy is to show that for some constant  $C$

$$\frac{d}{dt} H_N(t) \leq C H_N(t) + R_N(t) \quad (3.6)$$

with

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_0^t R_N(s) ds = 0. \quad (3.7)$$

Then it follows by Gronwall's inequality that  $\lim_{N \rightarrow \infty} \frac{H_N(t)}{N} = 0$  which concludes the proof of Theorem 3.1.

The following differential inequality comes from a straightforward calculation ([1, 5, 8]):

$$\frac{d}{dt} H_N(t) \leq - \int \left[ \left( N \mathcal{G}_N + \frac{\partial}{\partial t} \right) \log g_t^N \right] f_t^N d\mathbf{r} d\mathbf{p} \quad (3.8)$$

Before we proceed in the proof, we have to introduce some further notations:

For any  $C^1$  function  $F := (f_1, f_2, f_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  we define

$$DF(\mathbf{a}) := ((Df_1)(\mathbf{a}), (Df_2)(\mathbf{a}), (Df_3)(\mathbf{a}))^T,$$

with  $Df_i(\mathbf{a})$ ,  $i = 1, 2, 3$  defined by (2.12).

Recall that  $\tilde{\mathbf{u}} = (\mathbf{r}, \mathbf{p}, -E)$ ,  $\mathbf{e} := E - \frac{1}{2}\mathbf{p}^2$  and let us denote by

$$\tilde{\mathbf{J}}(\tilde{\mathbf{u}}) := (\mathbf{p}, P(\mathbf{r}, \mathbf{e}), -\mathbf{p}P(\mathbf{r}, \mathbf{e})) = \left( \mathbf{p}, \frac{\lambda_1(\mathbf{r}, \mathbf{e})}{\lambda_3(\mathbf{r}, \mathbf{e})}, -\mathbf{p} \frac{\lambda_1(\mathbf{r}, \mathbf{e})}{\lambda_3(\mathbf{r}, \mathbf{e})} \right) \quad (3.9)$$

the flux of (2.15), that can be rewritten as

$$\partial_t \tilde{\mathbf{u}} = D\tilde{\mathbf{J}}(\tilde{\mathbf{u}}) \partial_x \tilde{\mathbf{u}}$$



with the Jacobian

$$D\tilde{\mathbf{J}}(\tilde{\mathbf{u}}) \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial P}{\partial \mathbf{r}} & -\mathbf{p} \frac{\partial P}{\partial \mathbf{e}} & \frac{\partial P}{\partial \mathbf{e}} \\ -\mathbf{p} \frac{\partial P}{\partial \mathbf{r}} & -P + \mathbf{p}^2 \frac{\partial P}{\partial \mathbf{e}} & -\mathbf{p} \frac{\partial P}{\partial \mathbf{e}} \end{pmatrix} \quad (3.10)$$

With the dual relation (2.11),  $\boldsymbol{\lambda}$  is solution of the symmetric system

$$\partial_t[D\Theta(\boldsymbol{\lambda})] = \partial_x[D\Sigma(\boldsymbol{\lambda})], \quad (3.11)$$

where

$$\Sigma(\boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot \tilde{\mathbf{J}}(D\Theta(\boldsymbol{\lambda})).$$

Equation (3.11) can be rewritten as

$$(D^2\Theta)\partial_t\boldsymbol{\lambda} = (D^2\Sigma)\partial_x\boldsymbol{\lambda}.$$

Since

$$D^2\Theta(\boldsymbol{\lambda}(t, x))^{-1} = (D^2\Phi)(\tilde{\mathbf{u}}(t, x)),$$

it follows that

$$\partial_t\boldsymbol{\lambda}(D^2\Phi) = (D^2\Sigma)\partial_x\boldsymbol{\lambda}.$$

Since

$$(D^2\Sigma) = (D^2\Phi)(D\tilde{\mathbf{J}}(\tilde{\mathbf{u}}))$$

the following system of partial differential equations is satisfied:

$$\partial_t\boldsymbol{\lambda}(t, x) = (D\tilde{\mathbf{J}})^T(\tilde{\mathbf{u}})\partial_x\boldsymbol{\lambda}(t, x). \quad (3.12)$$

Let us define the microscopic fluxes:

$$\begin{aligned} \mathbf{J}_{i-1,i} &:= (-p_{i-1}, -V'(r_i), p_{i-1}V'(r_i)) \quad i = 1, \dots, N-1, \\ \mathbf{J}_{N,N+1} &:= (-p_N, -\tau(t), p_N\tau(t)) \end{aligned} \quad (3.13)$$

Then recalling the definition of the Liouville operator given by (2.4),

$$L_N^\tau \boldsymbol{\zeta}_i = \mathbf{J}_{i-1,i} - \mathbf{J}_{i,i+1}$$

Finally let us define

$$\mathbf{v}_j := (0, p_j, -p_j^2/2).$$

Hence with the definition of the symmetric operator given by (2.6),

$$\begin{aligned} S_N(\boldsymbol{\zeta}_j) &= -2\mathbf{v}_j + \mathbf{v}_{j+1} + \mathbf{v}_{j-1}, \quad j = 2, \dots, N-1 \\ S_N(\boldsymbol{\zeta}_N) &= -\mathbf{v}_N + \mathbf{v}_{N-1}, \quad S_N(\boldsymbol{\zeta}_1) = -\mathbf{v}_1 + \mathbf{v}_2 \end{aligned}$$

**Lemma 3.2.**

$$NL_N^\tau \log g_t^N = \sum_{i=1}^N \partial_x \boldsymbol{\lambda}(\frac{i}{N}, t) \cdot \mathbf{J}_{i-1,i} + N\lambda_2(1, t)\tau(t) + a_N(t) \quad (3.14)$$

where  $a_N(t)$  is such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_0^t \int a_N(s) f_s^N d\mathbf{p} d\mathbf{r} ds = 0$$

*Proof.*

$$\begin{aligned} NL_N^\tau \log g_t^N(\mathbf{r}, \mathbf{p}) &= N \sum_{i=1}^N \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \cdot (\mathbf{J}_{i-1,i} - \mathbf{J}_{i,i+1}) \\ &= N \sum_{i=1}^N \left( \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) - \boldsymbol{\lambda}\left(\frac{i-1}{N}, t\right) \right) \cdot \mathbf{J}_{i-1,i} - \boldsymbol{\lambda}(0, t) \cdot \mathbf{J}_{0,1} + \boldsymbol{\lambda}(1, t) \cdot \mathbf{J}_{N,N+1} \end{aligned}$$

Taking into account the boundary conditions on  $\boldsymbol{\lambda}$  we have

$$\boldsymbol{\lambda}(0, t) \cdot \mathbf{J}_{0,1} = \lambda_2(0, t) V'(r_1) = \mathbf{p}(0, t) V'(r_1) = 0 \quad (3.15)$$

and

$$\boldsymbol{\lambda}(1, t) \cdot \mathbf{J}_{N,N+1} = -p_N \lambda_1(1, t) - \lambda_2(1, t) \tau(t) + \lambda_3(1, t) \tau(t) p_N = -\lambda_2(1, t) \tau(t) \quad (3.16)$$

because  $\tau(t) \lambda_3(1, t) = \lambda_1(1, t)$ . Since  $\boldsymbol{\lambda}$  is a  $C^2$ -functions, we obtain (3.14) with

$$|a_N(t)| = \frac{C}{N} \sum_{i=1}^{N-1} \|\mathbf{J}_i\|$$

It remains to show, that  $\lim_{N \rightarrow \infty} \int_0^t \int \frac{a_N(s)}{N} d\nu_s^N ds = 0$ . This is an easy consequence of Lemma 3.5.  $\square$

**Lemma 3.3.**

$$\partial_t \log g_t^N = \sum_{i=1}^N (D\tilde{\mathbf{J}})^T(\tilde{\mathbf{u}}(\frac{i}{N}, t)) \partial_x \boldsymbol{\lambda}(\frac{i}{N}, t) \cdot \left( \boldsymbol{\zeta}_i - \tilde{\mathbf{u}}(\frac{i}{N}, t) \right)$$

*Proof.*

$$\begin{aligned} \frac{\partial}{\partial t} \log g_t^N &= \frac{\partial}{\partial t} \sum_{i=1}^N \left( \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \cdot \boldsymbol{\zeta}_i - \Theta\left(\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\right) \right) \\ &= \sum_{i=1}^N \partial_t \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \cdot \left( \boldsymbol{\zeta}_i - D\Theta\left(\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\right) \right) \end{aligned}$$

By (2.11),  $D\Theta\left(\boldsymbol{\lambda}\left(\frac{i}{N}, t\right)\right) = \tilde{\mathbf{u}}(\frac{i}{N}, t)$ , and (3.12) the result follows.  $\square$

**Lemma 3.4.** Recall the definition of the symmetric operator given by (2.6).

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_0^t \int N S_N \log g_s^N d\nu_s^N ds \leq \lim_{N \rightarrow \infty} \frac{1}{\sigma N} \int_0^t H_N(s) ds,$$

where  $\sigma$  is a constant independent of  $N$  with  $0 < \sigma < \frac{\beta}{2}$ .

*Proof.*

$$\begin{aligned} &S_N \log g_s^N \\ &= \sum_{i=2}^{N-1} \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \cdot (\mathbf{v}_{i-1} - 2\mathbf{v}_i + \mathbf{v}_{i+1}) + \boldsymbol{\lambda}\left(\frac{1}{N}, t\right) \cdot (-\mathbf{v}_1 + \mathbf{v}_2) + \boldsymbol{\lambda}(1, t) \cdot (\mathbf{v}_{N-1} - \mathbf{v}_N) \\ &= \sum_{i=2}^{N-1} \left( \boldsymbol{\lambda}\left(\frac{i-1}{N}, t\right) - 2\boldsymbol{\lambda}\left(\frac{i}{N}, t\right) + \boldsymbol{\lambda}\left(\frac{i+1}{N}, t\right) \right) \cdot \mathbf{v}_i \\ &\quad + \left( \boldsymbol{\lambda}\left(\frac{2}{N}, t\right) - \boldsymbol{\lambda}\left(\frac{1}{N}, t\right) \right) \cdot \mathbf{v}_1 + \left( \boldsymbol{\lambda}\left(\frac{N-1}{N}, t\right) - \boldsymbol{\lambda}(1, t) \right) \cdot \mathbf{v}_N \end{aligned}$$

In Lemma 3.5 we will show that the expectation of  $\frac{1}{N} \sum_i \|\mathbf{v}_i\|$  is uniformly bounded for all  $N$  and hence, since  $\boldsymbol{\lambda}$  is in  $C^2$ , the first term vanishes in the limit as  $N \rightarrow \infty$ .

Recall that by the entropy inequality (3.5), for any smooth function  $J$  and any  $\sigma > 0$  we have for  $k \in \{1, \dots, N\}$ :

$$\frac{1}{N} \int p_k^2 d\nu_s^N \leq \frac{1}{N\sigma} \log \int e^{\sigma p_k^2} \nu_{\boldsymbol{\lambda}(\cdot, s)}^N + \frac{1}{N\sigma} H(s)$$

Since this inequality is true for any  $\sigma > 0$ , the integral on the right hand side of the inequality is bounded as long as  $\sigma < \frac{\beta}{2}$  and hence the first term vanishes as  $N \rightarrow \infty$ . The expected value of  $p_k$  can be controlled in a similar way.  $\square$

To conclude the proofs of Lemma 3.2 and Lemma 3.4, it remains that the expected values of the densities are uniformly bounded:

**Lemma 3.5.**

$$\int \frac{1}{N} \sum_{i=1}^N e_i d\nu_t^N \leq C,$$

where  $C$  is a constant not depending on  $N$ .

*Proof.* By the conservation of total energy and our assumption on the initial state, we know that

$$\int \sum_{i=1}^N e_i d\nu_t^N = \int \sum_{i=1}^N e_i d\nu_0^N \leq CN.$$

$\square$

So far we have from Lemma 3.2, 3.3 and 3.4

$$\begin{aligned} \frac{d}{dt} H_N(t) &\leq \int \sum_{i=1}^N \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \left[ \mathbf{J}_{i-1, i} - (D\tilde{\mathbf{J}})^T(\tilde{\mathbf{u}}\left(\frac{i}{N}, t\right)) \left( \boldsymbol{\zeta}_i - \tilde{\mathbf{u}}\left(\frac{i}{N}, t\right) \right) \right] d\nu_t^N \\ &\quad - N\tau(t)\lambda_2(1, t) + \frac{1}{\sigma} H_N(t) + R_N(t) \end{aligned} \quad (3.17)$$

where  $R_N(t)$  is such that (3.7) holds.

With (3.9) we furthermore have

$$\int_0^1 \partial_x \boldsymbol{\lambda}(x, t) \cdot \tilde{\mathbf{J}}(\tilde{\mathbf{u}}(x, t)) dx = \int \frac{\partial}{\partial x} \left( \frac{\lambda_1(x, t)\lambda_2(x, t)}{\lambda_3(x, t)} \right) dx = \tau(t)\lambda_2(1, t).$$

This means that we can replace  $-N\tau(t)\lambda_2(1, t)$  by

$$-\sum_{i=1}^N \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \cdot \tilde{\mathbf{J}}\left(\tilde{\mathbf{u}}\left(\frac{i}{N}, t\right)\right)$$

with an error uniformly bounded in  $N$ , and consequently from (3.17) we have

$$\begin{aligned} \frac{d}{dt} H_N(t) &\leq \int \sum_{i=1}^N \partial_x \boldsymbol{\lambda}\left(\frac{i}{N}, t\right) \times \\ &\quad \left[ \mathbf{J}_{i-1, i} - \tilde{\mathbf{J}}\left(\tilde{\mathbf{u}}\left(\frac{i}{N}, t\right)\right) - (D\tilde{\mathbf{J}})^T(\tilde{\mathbf{u}}\left(\frac{i}{N}, t\right)) \left( \boldsymbol{\zeta}_i - \tilde{\mathbf{u}}\left(\frac{i}{N}, t\right) \right) \right] d\nu_t^N \\ &\quad + \frac{1}{\sigma} H_N(t) + R_N(t). \end{aligned} \quad (3.18)$$

Our next goal is to prove a weak form of local equilibrium. In view of this we introduce microscopic averages over blocks of size  $k+1$ : In what follows, for any

vectorfield  $\mathbf{Y}_i := (Y_{1,i}, Y_{2,i}, Y_{3,i}) : (\mathbb{R}^2)^3 \rightarrow \mathbb{R}^3$  we denote by  $\mathbf{Y}_i^k := (Y_{1,i}^k, Y_{2,i}^k, Y_{3,i}^k)$ , block averages over blocks of length  $k+1$ , where  $k > 0$  is independent of  $N$ . For example

$$\zeta_i^k = (\zeta_{1,i}^k, \zeta_{2,i}^k, \zeta_{3,i}^k) := (r_i^k, p_i^k, -e_i^k) := \frac{1}{k+1} \sum_{|i-l| \leq \frac{k}{2}} \zeta_l. \quad (3.19)$$

These blocks are microscopically large but on the macroscopic scale they are small, thus  $N$  goes to infinity first and then  $k$  goes to infinity.

For any smooth and bounded function  $J : [0, 1] \rightarrow \mathbb{R}$  and any bounded function  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$\frac{1}{N} \sum_{i=1}^N J\left(\frac{i}{N}\right) \psi(r_i, p_i) = \frac{1}{N} \sum_{i=\frac{k}{2}+1}^{N-\frac{k}{2}} J\left(\frac{i}{N}\right) \frac{1}{k+1} \sum_{|j-i| \leq \frac{k}{2}} \psi(r_j, p_j) + \mathcal{O}\left(\frac{k}{N}\right). \quad (3.20)$$

To apply the summation by parts formula to (3.18), we therefore first need to do some cut off in order to have only bounded variables: Let  $\mathcal{C}_{i,b} := \{|e_i| \leq b\}$ , and define

$$\mathbf{J}_{i-1,i}^b := \mathbf{J}_{i-1,i} \mathbf{1}_{\mathcal{C}_{i,b}} \quad \text{and} \quad \zeta_i^b := \zeta_i \mathbf{1}_{\mathcal{C}_{i,b}},$$

then these functions are bounded.

Assumptions (2.1) and (2.2) on the potential assert that by the entropy inequality (3.5) with reference measure  $d\nu_{\lambda(\cdot,t)}^N$ , the error we make by the replacement of  $\mathbf{J}_{i-1,i}$  and  $\zeta_i$  by  $\mathbf{J}_{i-1,i}^b$  and  $\zeta_i^b$  respectively is small in  $N$  if we can show that  $\frac{1}{N} H_N(s) \rightarrow 0$  as  $N \rightarrow 0$ :

For any  $\sigma > 0$  small enough

$$\begin{aligned} & \int \sum_{i=1}^N \partial_x \lambda\left(\frac{i}{N}, s\right) \mathbf{J}_{i-1,i} \mathbf{1}_{\mathcal{C}_{i,b}^c} d\nu_s^N \\ & \leq \frac{1}{\sigma} \sum_{i=1}^N \log \left( \int e^{\sigma \partial_x \lambda\left(\frac{i}{N}, s\right) \mathbf{J}_{i-1,i} \mathbf{1}_{\mathcal{C}_{i,b}^c} + \lambda\left(\frac{i}{N}, s\right) \zeta_i - \Theta\left(\lambda\left(\frac{i}{N}, s\right)\right)} d\nu_{\lambda(\cdot,t)} \right) + \frac{H_N(s)}{\sigma} \\ & \leq \frac{1}{\sigma} \sum_{i=1}^N \log \left( 1 + \int_{\mathcal{C}_{i,b}^c} e^{\sigma \partial_x \lambda\left(\frac{i}{N}, s\right) \mathbf{J}_{i-1,i} + \lambda\left(\frac{i}{N}, s\right) \zeta_i - \Theta\left(\lambda\left(\frac{i}{N}, s\right)\right)} d\nu_{\lambda(\cdot,t)} \right) + \frac{H_N(s)}{\sigma} \\ & = \frac{NC(b)}{\sigma} + \frac{H_N(s)}{\sigma} \end{aligned}$$

where  $\lim_{b \rightarrow \infty} C(b) = 0$ .

Using that of  $\lambda$  and  $\mathbf{u}$  are in  $C^2$  and the summation by parts formula (3.20), we arrive at

$$\begin{aligned} & \sum_{i=1}^N \partial_x \lambda\left(\frac{i}{N}, s\right) \left[ \mathbf{J}_{i-1,i}^b - \tilde{\mathbf{J}} \left( \mathbf{u}\left(\frac{i}{N}, s\right) \right) - (D\tilde{\mathbf{J}})^T \left( \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right) \right) \left( \zeta_i^b - \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right) \right) \right] \\ & = \sum_{i=\frac{k}{2}+1}^{N-\frac{k}{2}} \partial_x \lambda\left(\frac{i}{N}, s\right) \times \\ & \quad \left[ \frac{1}{k+1} \sum_{|l-i| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b - \tilde{\mathbf{J}} \left( \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right) \right) - (D\tilde{\mathbf{J}})^T \left( \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right) \right) \left( \zeta_i^{b,k} - \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right) \right) \right] + \mathcal{O}(k) \end{aligned}$$

since by the cut off,  $|\mathbf{J}_{i-1,i}^b - \zeta_i^b|$  is bounded.

We now introduce further block averages:  
For some small  $\ell > 0$ , such that  $\ell \rightarrow 0$  after  $N \rightarrow \infty$  and  $\ell N \gg k$ , we restrict the sum in the last expression as follows:

$$\begin{aligned} & \sum_{i=[N\ell]}^{N-[N\ell]} \partial_x \lambda\left(\frac{i}{N}, s\right) \times \\ & \left[ \frac{1}{k+1} \sum_{|l-i| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b - \tilde{\mathbf{J}}\left(\mathbf{u}\left(\frac{i}{N}, s\right)\right) - (D\tilde{\mathbf{J}})^T\left(\tilde{\mathbf{u}}\left(\frac{i}{N}, s\right)\right) \left(\boldsymbol{\zeta}_i^{b,k} - \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right)\right) \right] \\ & + C(k + N\ell), \end{aligned}$$

where  $C$  is a constant not depending on  $N, k$  and  $\ell$ . The error we made divided by  $N$  will vanish in the limit as well since  $\ell \rightarrow 0$ .

In what follows, we denote by  $\Lambda_i^{2n}$  the set of configurations in the box  $\{i-n, \dots, i+n\}$  and by  $\Lambda^m$  the set of configurations in any box of size  $m+1$ .

Let  $0 < \varepsilon < \ell$  small such  $\varepsilon N \rightarrow \infty$  as  $N \rightarrow \infty$ . To simplify the notation, we introduce

$$\begin{aligned} \Lambda^* &:= \Lambda^{N-2[N\ell]+k}, \\ \Lambda^{**} &:= \Lambda^{2[\varepsilon N]+k}, \\ \Lambda_i^{**} &:= \Lambda_i^{2[\varepsilon N]+k}, \\ \Lambda^{***} &:= \Lambda^{N-2[N\ell]+4[N\varepsilon]+k}. \end{aligned}$$

Performing again a summation by parts over blocks of size  $2[\varepsilon N] + 1$ , we get

$$\begin{aligned} & \frac{1}{N} \int_0^t \int \sum_{i=[N\ell]}^{N-[N\ell]} \partial_x \lambda\left(\frac{i}{N}, s\right) \tau_i \left( \frac{1}{k+1} \sum_{|l| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b \right) f_s^N d\mathbf{r} d\mathbf{p}|_{\Lambda^*} ds \\ &= \frac{1}{N} \int_0^t \sum_{i=[N\ell]}^{N-[N\ell]} \partial_x \lambda\left(\frac{i}{N}, s\right) \\ & \quad \times \int \frac{1}{2[\varepsilon N] + 1} \sum_{j; |j-i| \leq [\varepsilon N]} \tau_j \left( \frac{1}{(k+1)} \sum_{l; |l| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b \right) f_s^N|_{\Lambda_i^{**}} d\mathbf{r} d\mathbf{p}|_{\Lambda^{**}} ds + C\varepsilon \\ &= \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \int_0^t \partial_x \lambda\left(\frac{i}{N}, s\right) \\ & \quad \times \left( \int \frac{1}{k+1} \sum_{l: |l-[N\ell]+[N\varepsilon]| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b \right) \frac{1}{2[\varepsilon N] + 1} \sum_{j; |j-i| \leq [\varepsilon N]} \tau_j f_s^N|_{\Lambda_i^{**}} d\mathbf{r} d\mathbf{p}|_{\Lambda^{***}} ds + C\varepsilon \\ &= t \int \partial_x \lambda\left(\frac{i}{N}, s\right) \left( \frac{1}{k+1} \sum_{l: |l-[N\ell]+[N\varepsilon]| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b \right) \\ & \quad \times \left( \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \frac{1}{t} \int_0^t \frac{1}{2[\varepsilon N] + 1} \sum_{j; |j-i| \leq [\varepsilon N]} \tau_j f_s^N|_{\Lambda_i^{**}} ds \right) d\mathbf{r} d\mathbf{p}|_{\Lambda^{***}} + C\varepsilon. \end{aligned}$$

Also here the prize is small in  $N$  because of the cut off.

Let denote by

$$\hat{\nu}_t^{N,\varepsilon,\ell}(d\mathbf{r}, d\mathbf{p}) := \hat{f}_t^{N,\varepsilon,\ell} d\mathbf{r} d\mathbf{p}|_{\Lambda^{***}} \quad (3.21)$$

the measure corresponding to the density

$$\hat{f}_t^{N,\varepsilon,\ell}(\mathbf{r}, \mathbf{p}) := \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \frac{1}{t} \int_0^t \frac{1}{2[\varepsilon N] + 1} \sum_{j: |j-i| \leq [\varepsilon N]} \tau_j f_s^N|_{\Lambda_i^{***}} ds, \quad (3.22)$$

which is a function of  $(r_i, p_i)_{i \in \{[N\ell]-2[N\varepsilon]-\frac{k}{2}, \dots, N-[N\ell]-2[N\varepsilon]+\frac{k}{2}\}}$ . Then the last expression reads as

$$t \int \partial_x \lambda\left(\frac{i}{N}, t\right) \int \frac{1}{k+1} \sum_{l: |l-[N\ell]+[N\varepsilon]| \leq \frac{k}{2}} \mathbf{J}_{l,b} d\hat{\nu}_t^{N,\varepsilon,\ell}.$$

Similar we can do the summation by parts over blocks of size  $2[\varepsilon N] + 1$  for the term containing the empirical densities:

$$\begin{aligned} & \frac{1}{N} \int_0^t \int \sum_{i=[N\ell]}^{N-[N\ell]} \partial_x \lambda\left(\frac{i}{N}, s\right) (D\mathbf{J})^T \left( \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right) \right) \tau_i \tilde{\zeta}_{[N\ell]-[N\varepsilon]}^{b,k} d\nu_s^N ds \\ &= t \int \partial_x \lambda\left(\frac{i}{N}, t\right) (D\tilde{\mathbf{J}})^T \left( \tilde{\mathbf{u}}\left(\frac{i}{N}, t\right) \right) \tilde{\zeta}_{[N\ell]-[N\varepsilon]}^k d\hat{\nu}_t^{N,\varepsilon,\ell} \end{aligned}$$

Notice also that

$$\tilde{\mathbf{J}}^b(\zeta_i^k) := \tilde{\mathbf{J}}(\zeta_i^k) \mathbf{1}\{e_l \leq b; |i-l| \leq \frac{k}{2}\} = \tilde{\mathbf{J}}(\zeta_i^{b,k})$$

The next Theorem will be proved in Section 3.3. Known as the one-block estimate, it allows to replace the averages of the functions over the microscopic blocks by a function of the average and it is a crucial step towards the proof of the hydrodynamic limit:

**Theorem 3.6** (The one-block estimate). *Let  $J : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$  be a function with continuous first derivative. Then*

$$\lim_{b \rightarrow \infty} \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \int \left| \frac{1}{k+1} \sum_{l: |l-[N\ell]-[N\varepsilon]| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b - \tilde{\mathbf{J}}^b(\zeta_{[N\ell]-[N\varepsilon]}^k) \right| d\hat{\nu}_t^{N,\varepsilon,\ell} = 0 \quad (3.23)$$

where  $\hat{\nu}_t^{N,\varepsilon,\ell}$  is defined by (3.21).

With this Theorem we obtain:

$$\begin{aligned} \lim_{N \rightarrow \infty} H_N(t) &\leq \lim_{b \rightarrow \infty} \lim_{\ell \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \left( tN \int \partial_x \lambda\left(\frac{i}{N}, s\right) \cdot \omega_b(\tilde{\zeta}_{[N\ell]-[N\varepsilon]}^k, \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right)) d\hat{\nu}_t^{N,\varepsilon,\ell} \right. \\ &\quad - tN \int \partial_s \lambda\left(\frac{i}{N}, s\right) \cdot \mathbf{w}(\zeta_{[N\ell]-[N\varepsilon]}^{b,k} \tilde{\mathbf{u}}\left(\frac{i}{N}, s\right)) d\hat{\nu}_t^{N,\varepsilon,\ell} \\ &\quad \left. + \int_0^t R_{N,k,\varepsilon,\ell,b}(s) ds + \int_0^t \frac{H_N(s)}{\sigma'} ds \right) \quad (3.24) \end{aligned}$$

for some  $\sigma' > 0$  and  $R_{N,k,\varepsilon,\ell,b}$  is such that

$$\lim_{b \rightarrow \infty} \lim_{\ell \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \int_0^t \frac{R_{N,k,\varepsilon,\ell,b}}{N} ds = 0.$$

Furthermore we defined

$$\omega_b(\mathbf{z}, \tilde{\mathbf{u}}) := \tilde{\mathbf{J}}^b(\mathbf{z}) - \tilde{\mathbf{J}}(\tilde{\mathbf{u}}) \quad \text{and} \quad \mathbf{w}(\mathbf{z}, \tilde{\mathbf{u}}) := (\mathbf{z} - \tilde{\mathbf{u}}).$$

To simplify the notation further, let  $\mathbf{\Omega}_b$  be as follows:

$$\mathbf{\Omega}_b(\mathfrak{z}, \tilde{\mathbf{u}}) := \partial_x \boldsymbol{\lambda} \cdot \boldsymbol{\omega}_b(\mathfrak{z}, \tilde{\mathbf{u}}) - \partial_s \boldsymbol{\lambda} \cdot \mathbf{w}(\mathfrak{z}, \tilde{\mathbf{u}}).$$

Hence

$$D_{\mathfrak{z}} \mathbf{\Omega}_b(\mathfrak{z}, \tilde{\mathbf{u}}) = (D\tilde{\mathbf{J}}^b)^T(\mathfrak{z}) \cdot \partial_x \boldsymbol{\lambda} - \partial_s \boldsymbol{\lambda} \quad (3.25)$$

is equal to zero if  $\mathfrak{z}$  is a solution of (3.12). Consequently:

$$\mathbf{\Omega}_b(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) = 0, \quad D_{\mathfrak{z}} \mathbf{\Omega}_b(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) = 0.$$

Rewritten in terms of  $\mathbf{\Omega}_b$  the right hand side of (3.24) is equal to

$$tN \int \mathbf{\Omega}_b(\zeta_{[N\ell]-[N\epsilon]}^k, \tilde{\mathbf{u}}(\frac{i}{N}, s)) d\hat{\nu}_t^{N,\epsilon,\ell} + \int_0^t R_{N,k,\epsilon,\ell,b}(s) ds + \int_0^t \frac{H_N(s)}{\sigma'} ds$$

Observe that the first term is equal to

$$\int_0^t \int \sum_{i=[N\ell]}^{N-[N\ell]} \mathbf{\Omega}_b(\zeta_i^k, \tilde{\mathbf{u}}(\frac{i}{N}, s)) d\nu_s^N ds + C\epsilon Nb,$$

where  $C$  is a constant independent of  $N, b$  and  $\epsilon$ . Applying the entropy inequality (3.5) on this expression, we obtain that for some  $\sigma > 0$  it is bounded above by

$$\frac{1}{\sigma} \int_0^t \log \int \exp \left\{ \sigma \sum_{i=[N\ell]}^{N-[N\ell]} \mathbf{\Omega}_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s)) \right\} d\nu_{\lambda(\cdot, t)}^N ds + \frac{1}{\sigma} \int_0^t H_N(s) ds. \quad (3.26)$$

Hence it remains to prove, that the first term of this expression is of order  $o(N)$ .

This will be done using the following special case of Varadhan's Lemma:

**Theorem 3.7** (Varadhan's Lemma). *Let  $\nu_{\lambda}^n$  be the product homogeneous measure with marginals  $\nu_{\lambda}$  given by (2.13) and with rate function  $I : \mathbb{R}^2 \times \mathbb{R}_- \rightarrow \mathbb{R}$  defined by*

$$I(\mathbf{x}) := \Phi(\mathbf{x}) - \mathbf{x} \cdot \boldsymbol{\lambda} + \Theta(\boldsymbol{\lambda}).$$

*Then for any bounded continuous function  $F$  on  $\mathbb{R}^2 \times \mathbb{R}_-$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \int e^{nF(\zeta)} d\nu_{\lambda}^n = \sup_{\mathbf{x}} \{F(\mathbf{x}) - I(\mathbf{x})\}$$

*Proof.* A proof of this Theorem can be adapted from [1, 5, 9] □

In order to apply this Theorem we arrange the sum in (3.26) as sums over disjoint blocks and then take advantage of the fact that the local Gibbs measures are product measures:

Assume without loss of generality that  $k+1$  divides  $N - 2[N\ell] - \frac{k}{2}$ , then

$$\sum_{i=[N\ell]}^{N-[N\ell]} \mathbf{\Omega}_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s)) = \sum_{j \in \{-\frac{k}{2}, \dots, \frac{k}{2}\}} \sum_{i \in B_{[N\ell]}} \tau_j \mathbf{\Omega}_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s)), \quad (3.27)$$

where  $B_{[N\ell]} := \left\{ q(k+1) + [N\ell] + \frac{k}{2}; q \in \left\{ 0, \dots, \frac{N-2[N\ell]-\frac{k}{2}}{k+1} \right\} \right\}$ . In this way, for any fixed  $j$ , the terms in the sum over  $i \in B_{[N\ell]}$  depend on configurations in disjoint blocks, thus the random variables

$$\tau_j \mathbf{\Omega}_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s))$$

are independent under  $\nu_{\mathbf{u}(\cdot, s)}^N$  which is product. Consequently, if we apply the Hölder inequality, we obtain

$$\begin{aligned}
& \frac{1}{\sigma} \int_0^t \log \int \prod_{j \in \{-\frac{k}{2}, \dots, \frac{k}{2}\}} \exp \left\{ \sigma \sum_{i \in B_{[N\ell]}} \tau_j \Omega_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s)) \right\} d\nu_{\lambda(\cdot, s)}^N ds \\
& \leq \frac{1}{\sigma(k+1)} \int_0^t \sum_{j \in \{-\frac{k}{2}, \dots, \frac{k}{2}\}} \log \int \exp \left\{ \sigma(k+1) \sum_{i \in B_{[N\ell]}} \tau_j \Omega_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s)) \right\} d\nu_{\lambda_{\mathbf{u}(\cdot, s)}^N} ds \\
& = \frac{1}{\sigma(k+1)} \sum_{i=[N\ell]}^{N-[N\ell]} \int_0^t \log \int \exp \left\{ \sigma(k+1) \Omega_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s)) \right\} d\nu_{\lambda(\cdot, s)}^N d\mathbf{r} d\mathbf{p} |_{\{-\frac{k}{2}, \dots, \frac{k}{2}\}} ds.
\end{aligned}$$

Then, since all the functions in this expression are smooth and the family of local Gibbs measures converges weakly, we obtain that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{(k+1)N\sigma} \sum_{i=[N\ell]}^{N-[N\ell]} \int_0^t \log \int \exp \left\{ \sigma(k+1) \Omega_b(\zeta_i^k, \mathbf{u}(\frac{i}{N}, s)) \right\} d\nu_{\lambda(\cdot, s)}^N ds \\
& = \lim_{k \rightarrow \infty} \frac{1}{\sigma(k+1)} \int_0^t \int_0^1 \log \int \exp \left\{ (k+1) \sigma \Omega_b(\zeta_i^k, \mathbf{u}(x, s)) \right\} d\nu_{\lambda(x, s)}^k dx ds.
\end{aligned}$$

So now for each  $x \in [0, 1]$ , the distribution of the particles in a box of size  $k$  is given by the invariant Gibbs measure with average  $\mathbf{u}(x, s)$ . such that we can apply Theorem 3.7 on this product measure to obtain that the last expression is equal to

$$\frac{1}{\sigma} \int_0^t \int_0^1 \sup_{\mathfrak{z}} \{ \sigma \Omega_b(\mathfrak{z}, \mathbf{u}(x, s)) - I(\mathfrak{z}) \} dx. \quad (3.28)$$

To conclude Theorem 3.1 it thus remains to show that this is equal to zero. Since  $I$  and  $\Omega_b$  are both convex, since both functions and their derivatives are vanishing at  $\mathfrak{z} = \mathbf{u}$ , it follows from assumption (2.1) and (2.2) on the potential that  $\sigma \Omega_b(\mathfrak{z}, \mathbf{u}) \leq I(\mathfrak{z})$  for  $\sigma$  small enough. Hence there exists a  $\sigma$  such that the last expression is equal to zero.

This concludes the proof of Theorem 3.1:

Since

$$\frac{d}{dt} H_N(t) \leq C H_N(t) + R_{N, k, \varepsilon, \ell, b}(t),$$

for some uniform constant  $C$ , it follows by Gronwall inequality that

$$\begin{aligned}
H_N(t) & \leq H_N(0) e^{Ct} + \int_0^t R_{N, k, \varepsilon, \ell, b}(s) e^{C(t-s)} ds \\
& \leq e^{Ct} \left( H_N(0) + \int_0^t R_{N, k, \varepsilon, \ell, b}(s) ds \right).
\end{aligned}$$

Hence the claim follows, since

$$\lim_{b \rightarrow \infty} \lim_{\ell \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \int_0^t \frac{R_{N, k, \varepsilon, \ell, b}(s)}{N} ds = 0.$$

### 3.3 The one block estimate (Theorem 3.6)

#### 3.3.1 Tightness

For a fixed  $k > 0$  and  $i \in \{[N\ell], \dots, N[1-\ell]\}$ , let

$$\hat{\nu}_{t, i}^{N, \varepsilon, k}(d\mathbf{r}, d\mathbf{p}) := \hat{f}_{t, i}^{N, \varepsilon, k} \prod_{i=l-\frac{k}{2}-1}^{l+\frac{k}{2}} dr_i dp_i$$



be the projection on the configurations in a block of size  $k$  around site  $i$  of the measure  $\hat{\nu}_t^{N,\varepsilon,\ell}$  defined by (3.21). The density corresponding to  $\hat{\nu}_{t,i}^{N,\varepsilon,k}$ , is given by

$$\hat{f}_{t,i}^{N,\varepsilon,k} := \hat{f}_t^{N,\varepsilon,\ell}|_{\{i-\frac{k}{2}-1, \dots, i+\frac{k}{2}\}}. \quad (3.29)$$

We have the following

**Lemma 3.8** (Tightness). *For each  $k \geq 2$  fixed, the sequence  $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_{N \geq 1}$  of probability measures is tight.*

*Proof.* We need to prove that for each  $n > 0$ , the expected value

$$E_{\hat{\nu}_{t,i}^{N,\varepsilon,k}} \left\{ \frac{1}{k+1} \sum_{|i-l| \leq \frac{k}{2}} |e_l| > n \right\} \leq C \quad (3.30)$$

for a constant  $C$  independent of  $N$  (and  $n$ ).

Since this expectation depends on configurations only through  $l \in \{i-\frac{k}{2}, \dots, i+\frac{k}{2}\}$ , we can write

$$\int \frac{1}{k+1} \sum_{|i-l| \leq \frac{k}{2}} e_l \hat{f}_{t,i}^{N,\varepsilon,k} d\mathbf{r} d\mathbf{p} = \int \frac{1}{k+1} \sum_{|i-l| \leq \frac{k}{2}} e_l \hat{f}_t^{N,\varepsilon,\ell} d\mathbf{r} d\mathbf{p}. \quad (3.31)$$

In the following we define by  $\bar{\nu}_t^N := \bar{f}_t^N \prod_{i=1}^N d\mathbf{r} d\mathbf{p}$  the probability measure with density  $\bar{f}_t^N$  defined by the time average

$$\bar{f}_t^N(d\mathbf{r}, d\mathbf{p}) := \frac{1}{t} \int_0^t f_s^N ds \quad (3.32)$$

Using the definition (3.3) of  $\hat{f}_t^{N,\varepsilon,\ell}$ , the right hand side of (3.31) is equal to

$$\begin{aligned} & \int \left( \frac{1}{k+1} \sum_{l; |i-l| \leq \frac{k}{2}} e_l \right) \left( \frac{1}{N} \sum_{m=[N\ell]}^{N-[N\ell]} \frac{1}{2[\varepsilon N] + 1} \sum_{j; |j-m| \leq \varepsilon N} \tau_j \bar{f}_t^N|_{\Lambda_m^{2\varepsilon N+k}} \right) d\mathbf{r} d\mathbf{p} \\ &= \int \frac{1}{N} \sum_{m=[N\ell]}^{N-[N\ell]} \frac{1}{2[\varepsilon N] + 1} \sum_{j; |j-m| \leq \varepsilon N} \tau_j \left( \frac{1}{k+1} \sum_{l; |l-m| \leq \frac{k}{2}} e_l \right) \bar{f}_t^N|_{\Lambda_m^{2\varepsilon N+k}} d\mathbf{r} d\mathbf{p} \\ &= \frac{1}{N} \sum_{m=[N\ell]}^{N-[N\ell]} \int \frac{1}{2[\varepsilon N] + 1} \sum_{j; |j-m| \leq \varepsilon N} \tau_j \left( \frac{1}{k+1} \sum_{l; |l-i| \leq \frac{k}{2}} e_l \right) \bar{f}_t^N d\mathbf{r} d\mathbf{p} \\ &\leq C_1 \int \left( \frac{1}{N} \sum_{m=[N(\ell-\varepsilon)] - \frac{k}{2}}^{[N(1-\ell+\varepsilon)] + \frac{k}{2}} e_m \right) \bar{f}_t^N d\mathbf{r} d\mathbf{p} \end{aligned}$$

where  $C_1$  is a constant independent of  $N$ . This inequality is true, since the last expression averages out all the  $e_m$  from for  $m \in \{[N(\ell-\varepsilon)] - \frac{k}{2}, \dots, [N(1-\ell+\varepsilon)] + \frac{k}{2}\}$ . Hence this last expression is bounded above by the total energy which again is uniformly bounded (see proof of Lemma 3.5). This concludes the proof of (3.30).  $\square$

Lemma 3.8 asserts that for each fixed  $k \geq 2$  and each fixed  $i$  there exists a limit point  $\nu_{t,i}^{\varepsilon,k}$  of the sequence  $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_{N \geq 1}$ . On the other hand, since the sequence  $(\nu_{t,i}^{\varepsilon,k})_{k \geq 2}$  forms a consistent family of measures, by Kolmogorov's Theorem, for  $k \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , there exists a unique probability measure  $\nu$  on the configuration space  $\{(r_i, p_i)_{i \in \mathbb{Z}} \in (\mathbb{R}^2)^\infty\}$ , such that the restriction of  $\nu$  on  $\{(r_j, p_j)_{j \in \{i-\frac{k}{2}, \dots, i+\frac{k}{2}\}} \in (\mathbb{R}^2)^{k+1}\}$  is  $\nu_{t,i}^{k,\varepsilon}$ .

### 3.3.2 Proof of the one-block-estimate

Let us define the formal generator  $\mathcal{G}$  of the infinite dynamics by

$$\mathcal{G} := \mathcal{L} + \gamma \mathcal{S}, \quad (3.33)$$

with the antisymmetric part

$$\mathcal{L} := \sum_{j \in \mathbb{Z}} \left\{ p_j \left( \frac{\partial}{\partial r_j} - \frac{\partial}{\partial r_{j+1}} \right) + (V'(r_{j+1}) - V'(r_j)) \frac{\partial}{\partial p_j} \right\} \quad (3.34)$$

and the symmetric part

$$\mathcal{S} := \sum_{i \in \mathbb{Z}} \left( f(\mathbf{r}, \mathbf{p}^{j,j+1}) - f(\mathbf{r}, \mathbf{p}) \right) \quad (3.35)$$

In section 3.4 we will prove the following Proposition:

**Proposition 3.9.** *Any limit point  $\nu$  of  $\hat{\nu}_{t,i}^{N,\varepsilon,k}$  satisfies the following properties:*

- (i) *it has finite density entropy: there exists a constant  $C > 0$  such that for all subsets  $\Lambda \subset \mathbb{Z}$*

$$H_{|\Lambda|} \left( \mu|_{\Lambda} \left| \nu_{(\tau_{\beta,0,\beta})}^{|\Lambda|} \right. \right) \leq C|\Lambda|,$$

- (ii) *it is translation invariant: For any local function  $\psi$  and any  $j \in \mathbb{Z}$ ,*

$$\int \psi \, d\mu = \int (\tau_j \psi) \, d\mu$$

*where  $\tau_j$  denotes the spatial shift by  $j$  on the configurations.*

- (iii) *it is stationary with respect to the operator  $\mathcal{G}$ : For any smooth bounded local function  $\psi$*

$$\int (\mathcal{G}\psi) \, d\mu = 0.$$

With this Proposition, we can apply the ergodic theorem from [2]:

**Theorem 3.10** (Ergodicity). *Any limit point  $\nu$  of  $\hat{\nu}_{t,i}^{N,\varepsilon,k}(\mathbf{dr}, \mathbf{dp})$  is a convex combination of Gibbs i.e there exists a probability measure  $\alpha(d\boldsymbol{\lambda})$  on  $\mathbb{R}^2 \times \mathbb{R}^+$  such that*

$$\nu(\mathbf{dr}, \mathbf{dp}) = \int \alpha(d\boldsymbol{\lambda}) \prod_{i \in \mathbb{Z}} \frac{1}{Z(\boldsymbol{\lambda})} e^{\lambda \zeta_i} dr_i dp_i.$$

The proof of Theorem 3.10 is contained in [2], see also Chapter 2 of [1] for more details. The idea of the proof is the following: With Proposition 3.9 one can prove that  $\nu$  is separately stationary for  $\mathcal{L}$  and  $\mathcal{S}$ . This implies that the distribution of momenta conditioned on position  $\nu(d\mathbf{p}|\mathbf{r})$  is exchangeable. It is here where we need the noise. For details of this result please see Chapter 2.3 of [1]. In Chapter 2.2 of the same notes [1] there can be found a detailed proof of how this implies 3.10.

With Lemma 3.8 and Theorem 3.10 it will turn out, that 3.6 is an application of the law of large numbers:

*Proof of Theorem 3.6:*

Let  $N^* := [N\ell] - [N\varepsilon]$ . The measure  $\hat{\nu}_t^{N,\varepsilon,\ell}$  in expression (3.23) can be replaced by  $\hat{\nu}_{t,N^*}^{N,\varepsilon,k}$  since the configurations inside the integral depend on the configurations only through  $N^* - \frac{k}{2} - 1, \dots, N^* + \frac{k}{2}$ . Thus it is enough to show that for each  $b$  and each  $\ell$

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \int \left| \frac{1}{k+1} \sum_{l: |l-N^*| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b - \tilde{\mathbf{J}}^b \left( \boldsymbol{\zeta}_{N^*}^k \right) \right| d\hat{\nu}_{t,N^*}^{N,\varepsilon,k} \leq 0.$$

By Lemma 3.8 and (ii), iii of Proposition 3.9, the left hand side can be rewritten as

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \int \left| \frac{1}{k+1} \sum_{|l-N^*| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b - \tilde{\mathbf{J}}^b(\zeta_{N^*}^k) \right| d\nu$$

By Theorem 3.10  $\nu$  is a convex combination of Gibbs measures. Taking the conditional expectation with respect to  $\zeta_{N^*}^k$  we are left to prove

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \left( \int \left| \frac{1}{k+1} \sum_{|l| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b - \tilde{\mathbf{J}}^b(\mathbf{z}) \right| \prod_{i \in \mathbb{Z}} e^{\lambda(\mathbf{z}) \zeta_i} dr_i dp_i \right) \alpha(d\lambda) = 0.$$

Because of the cut off the expression inside the integral is bounded hence applying the dominated convergence Theorem it remains to prove

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \int \left| \frac{1}{k+1} \sum_{|l| \leq \frac{k}{2}} \mathbf{J}_{l-1,l}^b - \tilde{\mathbf{J}}^b(\mathbf{z}) \right| \prod_{i \in \mathbb{Z}} e^{\lambda(\mathbf{z}) \zeta_i} dr_i dp_i = 0.$$

But this is just the law of large numbers and converges to 0 if  $k \rightarrow \infty$  and holds in the limit as  $k \rightarrow \infty$

□

### 3.4 Proof of Proposition 3.9

We start with some notations:

- By  $\nu_{t,i}^{N,n} := f_{t,i}^{N,n} \prod_{l=i-\frac{n}{2}-1}^{i+\frac{n}{2}} dr_l dp_l$  we denote the reduction of the measure  $\nu_t^N$  to the box  $\{i - \frac{n}{2} - 1, \dots, i + \frac{n}{2}\}$  around site  $i$  and of size  $n + 2$ . Thus its density  $f_{t,i}^{N,n}$  is given by

$$f_{t,i}^{N,n} := f_t^N|_{\{i-\frac{n}{2}-1, \dots, i+\frac{n}{2}\}}$$

- By  $\bar{\nu}_{t,i}^{N,n} := \bar{f}_{t,i}^{N,n} \prod_{l=i-\frac{n}{2}-1}^{i+\frac{n}{2}} dr_l dp_l$  we denote the reduction of the measure  $\bar{\nu}_t^N$ , defined in (3.32), to a box around site  $i$  and of size  $n + 2$  with density

$$\bar{f}_{t,i}^{N,n} := \frac{1}{t} \int f_{s,i}^{N,n} ds.$$

- Finally we define the density

$$\tilde{f}_{t,i}^{N,\varepsilon,n}(\mathbf{r}, \mathbf{p}) := \frac{1}{2[\varepsilon N] + 1} \sum_{|j-i| \leq \varepsilon N} \tau_j \bar{f}_{t,i}^{N,n}.$$

With these notations,  $\hat{f}_t^{N,\varepsilon,\ell}$  reads as:

$$\hat{f}_t^{N,\varepsilon,\ell}(\mathbf{r}, \mathbf{p}) = \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \tilde{f}_{t,i}^{N,\varepsilon,[\varepsilon N]+k}(d\mathbf{r}, d\mathbf{p}).$$

With the introduction of the local averages it is easy to show that the limit point  $\nu$  are translation invariant and stationary with respect to  $\mathcal{G}$ :

### 3.4.1 Translation invariant stationary measures

**Lemma 3.11.** *The measure  $\nu$  is translation invariant.*

*Proof.* Let  $\psi$  be a local function. Since  $(\hat{f}_{t,i}^{N,\varepsilon,k})_N$  is tight, we only need to prove that for each  $z$  we have

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \int (\psi - \tau_z \psi) \hat{f}_{t,i}^{N,\varepsilon,k} d\mathbf{r} d\mathbf{p} = 0$$

For a fixed  $i$  the integral is equal to:

$$\int (\psi - \tau_z \psi) \hat{f}_t^{N,\varepsilon,\ell} d\mathbf{r} d\mathbf{p} = \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \int (\psi - \tau_z \psi) \tilde{f}_{t,i}^{N,\varepsilon,\varepsilon N+k} (d\mathbf{r}, d\mathbf{p}).$$

Then

$$\begin{aligned} \int (\psi - \tau_z \psi) \tilde{f}_{t,i}^{N,\varepsilon,\varepsilon N+k} d\mathbf{r} d\mathbf{p} &= \frac{1}{2[\varepsilon N] + 1} \int (\psi - \tau_z \psi) \sum_{j; |i-j| \leq \varepsilon N} \tau_j \tilde{f}_{t,i}^{N,2\varepsilon N+k} d\mathbf{r} d\mathbf{p} \\ &= \frac{1}{2[\varepsilon N] + 1} \int \sum_{j; |i-j| \leq \varepsilon N} (\tau_j \psi - \tau_{j+z} \psi) \tilde{f}_{t,i}^{N,2\varepsilon N+k} d\mathbf{r} d\mathbf{p} \\ &= \frac{1}{2[\varepsilon N] + 1} \int \left( \sum_{j; |i-j| \leq N\varepsilon} \tau_j \psi - \sum_{j; |i-j+z| \leq N\varepsilon} \tau_j \psi \right) \tilde{f}_t^N d\mathbf{r} d\mathbf{p} \\ &= \mathcal{O}\left(\frac{z}{\varepsilon N}\right) \end{aligned}$$

converges to 0 as  $N$  approaches  $\infty$ . Thereby  $\nu$  is translation invariant.  $\square$

Next we prove that  $\nu$  is invariant with respect to the formal generator  $\mathcal{G}$  defined by (3.33):

**Lemma 3.12.** *The measure  $\nu$  is stationary in time with respect to the generator  $\mathcal{G} = \mathcal{L} + \gamma S$ , that means for any bounded smooth local function  $\psi(\mathbf{r}, \mathbf{p})$*

$$\int \mathcal{G}\psi d\nu = 0. \quad (3.36)$$

*Proof.* Let  $\psi$  be a local function. By the same arguments as in the proof of Lemma (3.11) it is enough to show that

$$\lim_{N \rightarrow \infty} \frac{1}{2[\varepsilon N] + 1} \int \sum_{j; |i-j| \leq N\varepsilon} \tau_j (\mathcal{G}\psi) \tilde{f}_{t,i}^{N,2\varepsilon N+k} d\mathbf{r} d\mathbf{p} = 0. \quad (3.37)$$

For a fixed  $i$  define the spacial average  $\bar{\psi} := \frac{1}{2[\varepsilon N] + 1} \sum_{j; |i-j| \leq N\varepsilon} \tau_j \psi$ , then with

$$\mathcal{G}\bar{\psi} = \frac{1}{2[\varepsilon N] + 1} \sum_{j; |j-i| \leq N\varepsilon} \tau_j (\mathcal{G}\psi),$$

we can rewrite the integral of (3.37) as

$$\begin{aligned} \int (\mathcal{G}\bar{\psi}) \tilde{f}_{t,i}^{N,2\varepsilon N+k} d\mathbf{r} d\mathbf{p} &= \frac{1}{t} \int_0^t \int_0^s (\mathcal{G}\bar{\psi}) f_s^N d\mathbf{r} d\mathbf{p} ds \\ &= \frac{1}{tN} \int_0^t E_{\nu_s^N} \left[ \frac{\partial \bar{\psi}}{\partial s} \right] ds. \\ &= \frac{1}{tN} \left\{ E_{\nu_t^N} [\bar{\psi}] - E_{\nu_0^N} [\bar{\psi}] \right\}. \end{aligned}$$

We conclude the proof by observing that this expression converges to 0 if  $N \rightarrow \infty$ , since  $\psi$  and hence  $\bar{\psi}$  is a bounded function.  $\square$

### 3.4.2 Entropy density

Let

$$\nu_{(\tau\beta,0,\beta)}(d\mathbf{r}, d\mathbf{p}) := \prod_{i \in \mathbb{Z}} \nu_{(\tau\beta,0,\beta)}(dr_i, dp_i)$$

and

$$H_{\Lambda^k}(\nu | \nu_{(\tau\beta,0,\beta)}^\infty) := H(\nu |_{\Lambda^k} | \nu_{(\tau\beta,0,\beta)}^k) := H(\nu^k | \nu_{(\tau\beta,0,\beta)}^k).$$

Then we obtain the following Lemma:

**Lemma 3.13.** *The limit point  $\nu$  has finite entropy density, that means there exists a constant  $C > 0$  such that for all subsets  $\Lambda^k$*

$$H(\nu^k | \nu_{(\tau\beta,0,\beta)}^k) \leq C |\Lambda^k|.$$

In particular there exists the limit

$$\bar{H}(\nu | \nu_{\tau\beta,0,\beta}) = \lim_{k \rightarrow \infty} \frac{1}{k+1} H(\nu^k | \nu_{(\tau\beta,0,\beta)}^k) = \sup_k H(\nu^k | \nu_{(\tau,0,\beta)}^k).$$

*Proof.* By Lemma 3.8 the sequence  $(\hat{\nu}_{t,i}^{N,\varepsilon,k})_N$  is tight. By the lower semicontinuity of the relative entropy we have

$$\lim_{N \rightarrow \infty} H_k \left( \hat{\nu}_{t,i}^{N,\varepsilon,k} \middle| \nu_{(\tau\beta,0,\beta)}^k \right) = H_k \left( \nu_i^{\varepsilon,k} \middle| \nu_{(\tau\beta,0,\beta)}^k \right),$$

where for each  $i$ , the limit point  $\nu_i^{\varepsilon,k}$  is the restriction of  $\nu$  to the box  $\Lambda_i^k$ . Consequently  $\nu_i^{\varepsilon,k} = \nu|_{\Lambda_i^k} = \nu^k$  is translation invariant since by Lemma 3.11  $\nu$  is translation invariant. Hence

$$H_k \left( \nu^k \middle| \nu_{(\tau\beta,0,\beta)}^k \right) = H_{\Lambda^k}(\nu | \nu_{(\tau\beta,0,\beta)}).$$

One can prove that  $H_N$  is superadditive in the following sense (see for example [1]):

$$H_N(\hat{\nu}_t^{N,\varepsilon,\ell} | \nu_{(\tau\beta,0,\beta)}^{N-2N^*+2k+1}) \geq \frac{N-2[N\ell]}{k} H_k \left( \hat{\nu}_{t,i}^{N,\varepsilon,k} \middle| \nu_{(\tau\beta,0,\beta)}^k \right). \quad (3.38)$$

On the other hand we will prove the following bound on entropy in Lemma 3.14:

$$H_N(\bar{\nu}_t^N | \nu_{(\tau\beta,0,\beta)}^N) \leq CN \quad (3.39)$$

for some constant  $C$  independent of  $N$ . By these two results we obtain

$$\begin{aligned} \frac{N-2[N\ell]}{k} H_k \left( \hat{\nu}_{t,i}^{N,\varepsilon,k} \middle| \nu_{(\tau\beta,0,\beta)}^k \right) &\leq \frac{1}{N} \sum_{i=[N\ell]}^{N-[N\ell]} \frac{1}{2[\varepsilon N] + 1} \sum_{j: |j-i| \leq [\varepsilon N]} H_{\Lambda_{i+j}^{2[\varepsilon N]+k}}(\bar{f}_t^N | \nu_{(\tau,0,\beta)}^N) \\ &\leq CN \cdot \frac{N-2[N\ell]}{N} \end{aligned}$$

and the Lemma follows.  $\square$

To complete the proof of Lemma 3.13 it remains to show (3.39) :

**Lemma 3.14.** *If  $\nu_t^N$  satisfies*

$$H_N \left( \bar{\nu}_0^N \middle| \nu_{(\tau,0,\beta)}^N \right) \leq CN$$

*for some uniform constant  $C > 0$ , then for any  $N \in \mathbb{N}$*

$$H_N \left( \bar{\nu}_t^N \middle| \nu_{(\tau\beta,0,\beta)}^N \right) \leq CN.$$

*Proof.* Denote by  $g_*^N$  the density of the invariant measure  $\nu_{(\tau\beta,0,\beta)}^N$  with respect to the Lebesgue measure. Then we obtain:

$$\begin{aligned} \frac{d}{dt} H_N \left( \nu_t^N | \nu_{(\tau,0,\beta)}^N \right) &= \int \frac{\partial f_t^N}{\partial t} \log \frac{f_t^N}{g_*^N} d\mathbf{r} d\mathbf{p} \\ &= \int f_t^N N \mathcal{G}_N \left( \frac{1}{g_*^N} \log \frac{f_t^N}{g_*^N} \right) d\nu_{(\tau\beta,0,\beta)}^N. \\ &= \int N L_N^\tau \frac{f_t^N}{g_*^N} d\nu_{(\tau\beta,0,\beta)}^N + \gamma N \int f_t^N(\mathbf{r}, \mathbf{p}) S_N \log f_t^N(\mathbf{r}, \mathbf{p}) d\mathbf{r} d\mathbf{p}. \end{aligned}$$

Since the reference measure is invariant with respect to  $\mathcal{G}_N$ , the first term of the last expression is equal to zero. Hence, with the elementary inequality

$$\alpha \log \frac{\beta}{\alpha} \leq 2\sqrt{\alpha}(\sqrt{\beta} - \sqrt{\alpha}),$$

we arrive at

$$\begin{aligned} \frac{d}{dt} H_N(\nu_t^N | \nu_{(\tau,0,\beta)}^N) &\leq \gamma N \int 2\sqrt{f_t^N(\mathbf{r}, \mathbf{p})} S_N \sqrt{f_t^N(\mathbf{r}, \mathbf{p})} d\mathbf{r} d\mathbf{p} \\ &= -4\gamma N \int \sum_{i=1}^{N-1} \left( \Upsilon_{i,i+1} \sqrt{f_t^N(\mathbf{r}, \mathbf{p})} \right)^2 d\mathbf{r} d\mathbf{p} \leq 0. \end{aligned}$$

The result follows by integrating in time and by the convexity of the relative entropy.  $\square$

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